7.1 - Definition of the Laplace Transform

Definition: A **transform** transforms one function to another function.

Examples of transforms
The derivative and the integral:

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and $\frac{d}{dx}(x^3) = 3x^2$ are transforms.

$\int \frac{1}{x}\, dx = \ln|x| + C$ is a transform.

Transforms possess the linearity properties, e.g.,
$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$

Definition 7.1.1: Let $f(t)$ be a function defined for $t \geq 0$. Then the integral $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ is said to be the **Laplace transform** of f, provided that the integral converges.

The Laplace transform possesses the linearity properties.

Example 1: Evaluate $\mathscr{L}\{1\}$.

Example 2: Evaluate $\mathscr{L}\{t\}$.

Example: Evaluate $\mathscr{L}\{4t - 7\}$

Example: Evaluate $\mathscr{L}\{\sin kt\}$

Theorem 7.1.1: Transforms of Some Basic Functions

$$\text{(a) } \mathscr{L}\{1\} = \frac{1}{s}$$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ n = 1,\ 2,\ 3,\ \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

Definition: A function f is said to be of **exponential order** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Theorem: Sufficient Conditions for Existence

If f is piecewise continuous on $[0,\ \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

Theorem: Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = e^{-2t-5}$

Example: Use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = t^2 - e^{-9t} + 5$

$f(t) = \cos^2 t$